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Optimality conditions for linear programming problems with fuzzy coefficients

Hsien-Chung Wu

Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 802, Taiwan

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Abstract

The optimality conditions for linear programming problems with fuzzy coefficients are derived in this paper. Two solution concepts are proposed by considering the orderings on the set of all fuzzy numbers. The solution concepts proposed in this paper will follow from the similar solution concept, called the nondominated solution, in the multiobjective programming problem. Under these settings, the optimality conditions will be naturally elicited.

Keywords: Fuzzy numbers; Nondominated solutions; (crisp) Fuzzy constraints

1. Introduction

The occurrence of fuzziness in the real world is inevitable owing to some unexpected situations. Therefore, imposing fuzziness upon conventional optimization problems becomes an interesting research topic. The collection of papers on fuzzy optimization edited by Słowiński [1] and Delgado et al. [2] gives the main stream of this topic. Lai and Hwang [3,4] also give an insightful survey. On the other hand, the book edited by Słowiński and Teghem [5] provides comparisons between fuzzy optimization and stochastic optimization for multiobjective programming problems.

Bellman and Zadeh [6] inspired the development of fuzzy optimization by providing the aggregation operators, which combined the fuzzy goals and fuzzy decision space. After this motivation and inspiration, there appeared a lot of articles dealing with fuzzy optimization problems. Some interesting articles are Buckley [7,8], Julien [9] and Luhandjula et al. [10] using possibility distribution, Herrera et al. [11] and Zimmermann [12,13] using fuzzified constraints and objective functions, Inuiguchi et al. [14,15] using modality measures, Tanaka and Asai [16] using fuzzy parameters, and Lee and Li [17–19] considering the de Novo programming problem.

The duality of the fuzzy linear programming problem was firstly studied by Rodder and Zimmermann [20] considering the economic interpretation of the dual variables. After that, many interesting results regarding the duality of the fuzzy linear programming problem was investigated by Bector et al. [21–23], Liu et al. [24], Ramík [25], Verdegay [26] and Wu [27]. In this paper, we investigate the optimality conditions for linear programming problems with fuzzy coefficients.

E-mail address: hcwu@nknucc.nknu.edu.tw.

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In Section 2, we introduce some basic properties and arithmetics of fuzzy numbers. In Section 3, we formulate two linear programming problems with fuzzy coefficients. One considers crisp (conventional) linear constraints, and the other considers fuzzy linear constraints. Two solution concepts are proposed for these two problems. In Section 4, we derive the optimality conditions for these two problems by introducing the multipliers. Finally, in Section 5, three examples are provided to illustrate the discussions in linear programming problems with fuzzy coefficients.

2. Arithmetics of fuzzy numbers

Let $\mathbb{R}$ be the set of all real numbers. The fuzzy subset $\tilde{a}$ of $\mathbb{R}$ is defined by a function $\xi_{\tilde{a}} : \mathbb{R} \to [0, 1]$, which is called a *membership function*. The α-level set of $\tilde{a}$, denoted by $\tilde{a}_\alpha$, is defined by $\tilde{a}_\alpha = \{x \in \mathbb{R} : \xi_{\tilde{a}}(x) \geq \alpha\}$ for all $\alpha \in (0, 1]$. The 0-level set $\tilde{a}_0$ is defined as the closure of the set $\{x \in \mathbb{R} : \xi_{\tilde{a}}(x) > 0\}$, i.e., $\tilde{a}_0 = \mathrm{cl}\,(\{x \in \mathbb{R} : \xi_{\tilde{a}}(x) > 0\})$.

Definition 2.1. We denote by $\mathcal{F}(\mathbb{R})$ the set of all fuzzy subsets $\tilde{a}$ of $\mathbb{R}$ with membership function $\xi_{\tilde{a}}$ satisfying the following conditions:

(i) $\tilde{a}$ is normal, i.e., there exists an $x \in \mathbb{R}$ such that $\xi_{\tilde{a}}(x) = 1$;
(ii) $\xi_{\tilde{a}}$ is quasi-concave, i.e., $\xi_{\tilde{a}}(\lambda x + (1 - \lambda)y) \geq \min\{\xi_{\tilde{a}}(x), \xi_{\tilde{a}}(y)\}$ for all $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$;
(iii) $\xi_{\tilde{a}}$ is upper semicontinuous, i.e., $\{x \in \mathbb{R} : \xi_{\tilde{a}}(x) \geq \alpha\} = \tilde{a}_\alpha$ is a closed subset of U for each $\alpha \in (0, 1]$;
(iv) the 0-level set $\tilde{a}_0$ is a compact subset of $\mathbb{R}$.

The member $\tilde{a}$ in $\mathcal{F}(\mathbb{R})$ is called a *fuzzy number*.

Suppose now that $\tilde{a} \in \mathcal{F}(\mathbb{R})$. From Zadeh [28], the α-level set $\tilde{a}_\alpha$ of $\tilde{a}$ is a convex subset of $\mathbb{R}$ for each $\alpha \in [0, 1]$ from condition (ii). Combining this fact with conditions (iii) and (iv), the α-level set $\tilde{a}_\alpha$ of $\tilde{a}$ is a compact and convex subset of $\mathbb{R}$ for each $\alpha \in [0, 1]$, i.e., $\tilde{a}_\alpha$ is a closed interval in $\mathbb{R}$ for each $\alpha \in [0, 1]$. Therefore, we also write $\tilde{a}_\alpha = \left[\tilde{a}^L_\alpha, \tilde{a}^U_\alpha\right]$.

Definition 2.2. Let $\tilde{a}$ be a fuzzy number. We say that $\tilde{a}$ is *nonnegative* if $\tilde{a}^L_\alpha \geq 0$ for all $\alpha \in [0, 1]$. We say that $\tilde{a}$ is *positive* if $\tilde{a}^L_\alpha > 0$ for all $\alpha \in [0, 1]$. We say that $\tilde{a}$ is *nonpositive* if $\tilde{a}^U_\alpha \leq 0$ for all $\alpha \in [0, 1]$. We say that $\tilde{a}$ is *negative* if $\tilde{a}^U_\alpha < 0$ for all $\alpha \in [0, 1]$.

Remark 2.1. Let $\tilde{a}$ be a fuzzy number. Then $\tilde{a}^L_\alpha \leq \tilde{a}^U_\alpha$ for all $\alpha \in [0, 1]$. Therefore if $\tilde{a}$ is nonnegative then $\tilde{a}^L_\alpha \geq 0$ and $\tilde{a}^U_\alpha \geq 0$ for all $\alpha \in [0, 1]$, and if $\tilde{a}$ is positive then $\tilde{a}^L_\alpha > 0$ and $\tilde{a}^U_\alpha > 0$ for all $\alpha \in [0, 1]$. We can have similar consequences for nonpositive and negative fuzzy numbers.

Let "$\odot$" be any binary operations $\oplus$ or $\otimes$ between two fuzzy numbers $\tilde{a}$ and $\tilde{b}$. The membership function of $\tilde{a} \odot \tilde{b}$ is defined by

$$\xi_{\tilde{a}\odot\tilde{b}}(z) = \sup_{x \circ y = z} \min\{\xi_{\tilde{a}}(x), \xi_{\tilde{b}}(y)\}$$

using the extension principle in Zadeh [29], where the operations $\odot = \oplus$ and $\otimes$ correspond to the operations $\circ = +$ and $\times$, respectively. Then we have the following results.

Proposition 2.1. *Let $\tilde{a}, \tilde{b} \in \mathcal{F}(\mathbb{R})$. Then we have*

(i) $\tilde{a} \oplus \tilde{b} \in \mathcal{F}(\mathbb{R})$ *and*

$$(\tilde{a} \oplus \tilde{b})_\alpha = \left[\tilde{a}^L_\alpha + \tilde{b}^L_\alpha, \tilde{a}^U_\alpha + \tilde{b}^U_\alpha\right];$$

(ii) $\tilde{a} \otimes \tilde{b} \in \mathcal{F}(\mathbb{R})$ *and*

$$(\tilde{a} \otimes \tilde{b})_\alpha = \left[\min\left\{\tilde{a}^L_\alpha \tilde{b}^L_\alpha, \tilde{a}^L_\alpha \tilde{b}^U_\alpha, \tilde{a}^U_\alpha \tilde{b}^L_\alpha, \tilde{a}^U_\alpha \tilde{b}^U_\alpha\right\}, \max\left\{\tilde{a}^L_\alpha \tilde{b}^L_\alpha, \tilde{a}^L_\alpha \tilde{b}^U_\alpha, \tilde{a}^U_\alpha \tilde{b}^L_\alpha, \tilde{a}^U_\alpha \tilde{b}^U_\alpha\right\}\right].$$

The following proposition is very useful for discussing the optimality conditions.

Proposition 2.2. *Let $\tilde{a}$ be a nonnegative fuzzy number and $\tilde{b}$ be a nonpositive fuzzy number. If $\tilde{a} \otimes \tilde{b} = \tilde{0}$, then $\tilde{a}^L_\alpha \tilde{b}^L_\alpha = \tilde{a}^L_\alpha \tilde{b}^U_\alpha = \tilde{a}^U_\alpha \tilde{b}^L_\alpha = \tilde{a}^U_\alpha \tilde{b}^U_\alpha = 0$ for all $\alpha \in [0, 1]$.*

Proof. From Proposition 2.1 (ii), we immediately have that $\tilde{a} \otimes \tilde{b} = \tilde{0}$ implies

$$\tilde{a}^U_\alpha \tilde{b}^L_\alpha = 0 = \tilde{a}^L_\alpha \tilde{b}^U_\alpha \tag{1}$$

for all $\alpha \in [0, 1]$. Therefore, we need to show that $\tilde{a}^L_\alpha \tilde{b}^L_\alpha = \tilde{a}^U_\alpha \tilde{b}^U_\alpha = 0$ for all $\alpha \in [0, 1]$. We consider the following cases.

(i) Suppose that $\tilde{b}^U_\alpha \neq 0$. Then $\tilde{b}^L_\alpha \neq 0$ since $\tilde{b}^L_\alpha \leq \tilde{b}^U_\alpha$. Therefore, $\tilde{a}^L_\alpha = \tilde{a}^U_\alpha = 0$ by (1), i.e., $\tilde{a}^L_\alpha \tilde{b}^L_\alpha = \tilde{a}^U_\alpha \tilde{b}^U_\alpha = 0$.
(ii) Suppose that $\tilde{b}^U_\alpha = 0$ and $\tilde{b}^L_\alpha = 0$. Then it is easy to see that $\tilde{a}^L_\alpha \tilde{b}^L_\alpha = \tilde{a}^U_\alpha \tilde{b}^U_\alpha = 0$.
(iii) Suppose that $\tilde{b}^U_\alpha = 0$ and $\tilde{b}^L_\alpha \neq 0$. Then it remains to show $\tilde{a}^L_\alpha \tilde{b}^L_\alpha = 0$. Since $\tilde{b}^L_\alpha \neq 0$ and $\tilde{a}^U_\alpha \tilde{b}^L_\alpha = 0$ by (1), we have $\tilde{a}^U_\alpha = 0$. Since $\tilde{a}$ is nonnegative, $0 \leq \tilde{a}^L_\alpha \leq \tilde{a}^U_\alpha = 0$ shows $\tilde{a}^L_\alpha \tilde{b}^L_\alpha = 0$. We complete the proof. ■

We say that $\tilde{a}$ is a *crisp number* with value m if its membership function is given by

$$\xi_{\tilde{a}}(r) = \begin{cases} 1 & \text{if } r = m \\ 0 & \text{otherwise.} \end{cases}$$

We also use the notation $\tilde{1}_{\{m\}}$ to represent the crisp number with value m. It is easy to see that $(\tilde{1}_{\{m\}})^L_\alpha = (\tilde{1}_{\{m\}})^U_\alpha = m$ for all $\alpha \in [0, 1]$. Let us remark that a real number m can be regarded as a crisp number $\tilde{1}_{\{m\}}$.

3. Solution concept

Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two closed intervals in $\mathbb{R}$. We write $B \leq A$ if and only if $b^L \leq a^L$ and $b^U \leq a^U$, and $B < A$ if and only if the following conditions are satisfied:

$$\begin{cases} b^L < a^L \\ b^U \leq a^U \end{cases} \text{ or } \begin{cases} b^L \leq a^L \\ b^U < a^U \end{cases} \text{ or } \begin{cases} b^L < a^L \\ b^U < a^U \end{cases}.$$

Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. Then $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ and $\tilde{b}_\alpha = [\tilde{b}^L_\alpha, \tilde{b}^U_\alpha]$ are two closed intervals in $\mathbb{R}$ for all $\alpha \in [0, 1]$. We write $\tilde{b} \preceq \tilde{a}$ if and only if $\tilde{b}_\alpha \leq \tilde{a}_\alpha$ for all $\alpha \in [0, 1]$, or equivalently, $\tilde{b}^L_\alpha \leq \tilde{a}^L_\alpha$ and $\tilde{b}^U_\alpha \leq \tilde{a}^U_\alpha$ for all $\alpha \in [0, 1]$. It is easy to see that "$\preceq$" is a partial ordering on $\mathcal{F}(\mathbb{R})$.

For notational convenience, we denote by $\tilde{0}$ the crisp number $\tilde{1}_{\{0\}}$ with value 0. Now we consider the following two linear programming problems with fuzzy coefficients:

$$\begin{array}{lll} \text{(FLP1)} & \min & (\tilde{c}_1 \otimes \tilde{1}_{x_1}) \oplus \cdots \oplus (\tilde{c}_n \otimes \tilde{1}_{x_n}) \\ & \text{subject to} & a_{j1}x_1 + \cdots + a_{jn}x_n + b_j \leq 0 \text{ for } j = 1, \ldots, m \\ & & x_1, \ldots, x_n \geq 0 \end{array}$$

and

$$\begin{array}{lll} \text{(FLP2)} & \min & (\tilde{c}_1 \otimes \tilde{1}_{x_1}) \oplus \cdots \oplus (\tilde{c}_n \otimes \tilde{1}_{x_n}) \\ & \text{subject to} & (\tilde{a}_{j1} \otimes \tilde{1}_{x_1}) \oplus \cdots \oplus (\tilde{a}_{jn} \otimes \tilde{1}_{x_n}) \oplus \tilde{b}_j \preceq \tilde{0} \text{ for } j = 1, \ldots, m \\ & & x_1, \ldots, x_n \geq 0 \end{array}$$

where $\tilde{1}_{\{x_i\}}$ is a crisp number with value x_i for $i = 1, \ldots, n$. Problem (FLP1) considers the crisp (conventional) constraints, and problem (FLP2) considers fuzzy constraints. For convenient presentation, we also write

$$\tilde{f}(\mathbf{x}) = (\tilde{c}_1 \otimes \tilde{1}_{x_1}) \oplus \cdots \oplus (\tilde{c}_n \otimes \tilde{1}_{x_n})$$
$$g_j(\mathbf{x}) = a_{j1}x_1 + \cdots + a_{jn}x_n + b_j$$
$$\tilde{g}_j(\mathbf{x}) = (\tilde{a}_{j1} \otimes \tilde{1}_{x_1}) \oplus \cdots \oplus (\tilde{a}_{jn} \otimes \tilde{1}_{x_n}) \oplus \tilde{b}_j.$$

We need to interpret the meaning of minimization in problems (FLP1) and (FLP2). Since "$\preceq$" is a partial ordering, not a total ordering, on $\mathcal{F}(\mathbb{R})$, we may follow the similar solution concept (the nondominated solution) used in multiobjective programming problem to interpret the meaning of minimization in problems (FLP1) and (FLP2).

Two types of nondominated solution will be considered. We write $\tilde{a} \prec_I \tilde{b}$ if and only if $\tilde{a}_\alpha \preceq \tilde{b}_\alpha$ for all $\alpha \in [0, 1]$ and there exists an $\alpha^* \in [0, 1]$ such that $\tilde{a}_{\alpha^*} \prec \tilde{b}_{\alpha^*}$, i.e.,

$$\begin{cases} \tilde{a}^L_{\alpha^*} < \tilde{b}^L_{\alpha^*} \\ \tilde{a}^U_{\alpha^*} \leq \tilde{b}^U_{\alpha^*} \end{cases} \text{or} \begin{cases} \tilde{a}^L_{\alpha^*} \leq \tilde{b}^L_{\alpha^*} \\ \tilde{a}^U_{\alpha^*} < \tilde{b}^U_{\alpha^*} \end{cases} \text{or} \begin{cases} \tilde{a}^L_{\alpha^*} < \tilde{b}^L_{\alpha^*} \\ \tilde{a}^U_{\alpha^*} < \tilde{b}^U_{\alpha^*} \end{cases}. \tag{2}$$

Therefore, we see that $\tilde{a} \prec_I \tilde{b}$ means $\tilde{a} \preceq \tilde{b}$ and $\tilde{a} \neq \tilde{b}$. On the other hand, we write $\tilde{a} \prec_{II} \tilde{b}$ if and only if $\tilde{a}_\alpha \prec \tilde{b}_\alpha$ for all $\alpha \in [0, 1]$, i.e., (2) is satisfied for each $\alpha \in [0, 1]$.

Definition 3.1. Let $\mathbf{x}^*$ be a feasible solution of problem (FLP1). We say that $\mathbf{x}^*$ is a *nondominated type-I solution* (resp. *nondominated type-II solution*) of problem (FLP1) if there exists no feasible solution $\bar{\mathbf{x}}$ such that $\tilde{f}(\bar{\mathbf{x}}) \prec_I \tilde{f}(\mathbf{x}^*)$ (resp. $\tilde{f}(\bar{\mathbf{x}}) \prec_{II} \tilde{f}(\mathbf{x}^*)$). The nondominated type-I and type-II solutions can be similarly considered for problem (FLP2).

In what follows, we are going to provide the optimality conditions for nondominated solution of problems (FLP1) and (FLP2).

4. The optimality conditions

In order to derive the optimality conditions of problems (FLP1) and (FLP2), we need to recall the Karush–Kuhn–Tucker conditions for nonlinear programming problem. Let f and g_j, $j = 1, \ldots, m$, be real-valued functions defined on $\mathbb{R}^n$. Then we consider the following (conventional) nonlinear programming problem:

$$\begin{aligned} \text{(NLP)} \quad & \min \quad f(\mathbf{x}) = f(x_1, \ldots, x_n) \\ & \text{subject to} \quad g_j(\mathbf{x}) \leq 0, \ j = 1, \ldots, m. \end{aligned}$$

Suppose that the constraint functions g_j are convex on $\mathbb{R}^n$ for each $j = 1, \ldots, m$. Then the well-known Karush–Kuhn–Tucker optimality conditions for problem (NLP) (e.g., see Horst et al. [30] or Bazarra et al. [31]) is stated below.

Theorem 4.1. *Assume that the constraint functions $g_j : \mathbb{R}^n \to \mathbb{R}$ are convex on $\mathbb{R}^n$ for $j = 1, \ldots, m$. Let $X = \{\mathbf{x} \in \mathbb{R}^n : g_j(\mathbf{x}) \leq 0, i = 1, \ldots, m\}$ be a feasible set and a point $\mathbf{x}^* \in X$. Suppose that the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is convex at $\mathbf{x}^*$, and f, g_j, $j = 1, \ldots, m$, are continuously differentiable at $\mathbf{x}^*$. If there exist (Lagrange) multipliers $0 \leq \mu_j \in \mathbb{R}$, $j = 1, \ldots, m$, such that*

(i) $\nabla f(\mathbf{x}^*) + \sum_{i=1}^m \mu_j \nabla g_j(\mathbf{x}^*) = \mathbf{0}$;
(ii) $\mu_j g_j(\mathbf{x}^*) = 0$ *for all* $j = 1, \ldots, m$,

then $\mathbf{x}^*$ is an optimal solution of problem (NLP).

Now, we are in a position to derive the optimality conditions for problems (FLP1) and (FLP2).

4.1. Crisp (conventional) constraints

Now we consider the problem (FLP1) with crisp constraints. We adopt the following notations:

$$\mathbf{c}^L_\alpha = \begin{bmatrix} \tilde{c}^L_{1\alpha} \\ \vdots \\ \tilde{c}^L_{n\alpha} \end{bmatrix}, \quad \mathbf{c}^U_\alpha = \begin{bmatrix} \tilde{c}^U_{1\alpha} \\ \vdots \\ \tilde{c}^U_{n\alpha} \end{bmatrix}, \quad \text{and} \quad \mathbf{a}_j = \begin{bmatrix} a_{j1} \\ \vdots \\ a_{jn} \end{bmatrix},$$

where $\tilde{c}^L_{i\alpha} = (\tilde{c}_i)^L_\alpha$ and $\tilde{c}^U_{i\alpha} = (\tilde{c}_i)^U_\alpha$ for $i = 1, \ldots, n$. We also see that $\tilde{c}^L_{i\alpha} \leq \tilde{c}^U_{i\alpha}$ for all $\alpha \in [0, 1]$ and all $i = 1, \ldots, m$. We also denote by $\mathbf{e}_k$ the unit vector in $\mathbb{R}^n$ for $k = 1, \ldots, n$, i.e., the kth component of $\mathbf{e}_k$ is 1 and the other components of $\mathbf{e}_k$ are zero.

Theorem 4.2. *Let $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ be a feasible solution of problem* (FLP1). *If there exist positive real-valued functions λ^L and λ^U defined on* $[0, 1]$, *and nonnegative real-valued functions μ_j and λ_k for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ defined on* $[0, 1]$ *such that the following conditions are satisfied:*

(i) $\lambda^L(\alpha) \cdot \mathbf{c}_\alpha^L + \lambda^U(\alpha) \cdot \mathbf{c}_\alpha^U + \sum_{j=1}^m \mu_j(\alpha) \cdot \mathbf{a}_j - \sum_{k=1}^n \lambda_k(\alpha) \cdot \mathbf{e}_k = \mathbf{0}$ *for all* $\alpha \in [0, 1]$;
(ii) $\mu_j(\alpha) \cdot g_j(\mathbf{x}^*) = 0 = \lambda_k(\alpha) \cdot x_k^*$ *for all* $\alpha \in [0, 1]$, *all* $j = 1, \ldots, m$ *and all* $k = 1, \ldots, n$, *where* $g_j(\mathbf{x}^*) = a_{j1}x_1^* + \cdots + a_{jn}x_n^* + b_j$,

then $\mathbf{x}^*$ *is a nondominated type-I solution of problem* (FLP1).

Proof. We are going to prove this result by contradiction. Suppose that conditions (i) and (ii) are satisfied and $\mathbf{x}^*$ is not a nondominated type-I solution. Then there exists a feasible solution $\bar{\mathbf{x}}$ such that $\tilde{f}(\bar{\mathbf{x}}) \prec_I \tilde{f}(\mathbf{x}^*)$, i.e., from (2),

$$\begin{cases} \tilde{f}_{\alpha^*}^L(\bar{\mathbf{x}}) < \tilde{f}_{\alpha^*}^L(\mathbf{x}^*) \\ \tilde{f}_{\alpha^*}^U(\bar{\mathbf{x}}) \leq \tilde{f}_{\alpha^*}^U(\mathbf{x}^*) \end{cases} \quad \text{or} \quad \begin{cases} \tilde{f}_{\alpha^*}^L(\bar{\mathbf{x}}) \leq \tilde{f}_{\alpha^*}^L(\mathbf{x}^*) \\ \tilde{f}_{\alpha^*}^U(\bar{\mathbf{x}}) < \tilde{f}_{\alpha^*}^U(\mathbf{x}^*) \end{cases} \quad \text{or} \quad \begin{cases} \tilde{f}_{\alpha^*}^L(\bar{\mathbf{x}}) < \tilde{f}_{\alpha^*}^L(\mathbf{x}^*) \\ \tilde{f}_{\alpha^*}^U(\bar{\mathbf{x}}) < \tilde{f}_{\alpha^*}^U(\mathbf{x}^*) \end{cases} \tag{3}$$

for some $\alpha^* \in [0, 1]$. Since $x_i \geq 0$ for $i = 1, \ldots, n$ and $\tilde{c}_{i\alpha}^L \leq \tilde{c}_{i\alpha}^U$ for all $\alpha \in [0, 1]$ and all $i = 1, \ldots, n$, using Proposition 2.1, we have

$$\tilde{f}_\alpha^L(\mathbf{x}) = \tilde{c}_{1\alpha}^L \cdot x_1 + \cdots + \tilde{c}_{n\alpha}^L \cdot x_n \quad \text{and} \quad \tilde{f}_\alpha^U(\mathbf{x}) = \tilde{c}_{1\alpha}^U \cdot x_1 + \cdots + \tilde{c}_{n\alpha}^U \cdot x_n. \tag{4}$$

We also have that

$$\nabla \tilde{f}_\alpha^L(\mathbf{x}) = \mathbf{c}_\alpha^L \quad \text{and} \quad \nabla \tilde{f}_\alpha^U(\mathbf{x}) = \mathbf{c}_\alpha^U. \tag{5}$$

We now define a real-valued function

$$f(\mathbf{x}) = \left(\lambda^L(\alpha^*) \cdot \tilde{c}_{1\alpha^*}^L + \lambda^U(\alpha^*) \cdot \tilde{c}_{1\alpha^*}^U\right) \cdot x_1 + \cdots + \left(\lambda^L(\alpha^*) \cdot \tilde{c}_{n\alpha^*}^L + \lambda^U(\alpha^*) \cdot \tilde{c}_{n\alpha^*}^U\right) \cdot x_n. \tag{6}$$

Then we see that

$$f(\mathbf{x}) = \lambda^L(\alpha^*) \cdot \tilde{f}_{\alpha^*}^L(\mathbf{x}) + \lambda^U(\alpha^*) \cdot \tilde{f}_{\alpha^*}^U(\mathbf{x}). \tag{7}$$

Combining (3) and (7), we see that

$$f(\bar{\mathbf{x}}) < f(\mathbf{x}^*) \tag{8}$$

since $\lambda^L(\alpha^*) > 0$ and $\lambda^U(\alpha^*) > 0$. Furthermore, from (5) and (7), we have

$$\nabla f(\mathbf{x}) = \lambda^L(\alpha^*) \cdot \nabla \tilde{f}_{\alpha^*}^L(\mathbf{x}) + \lambda^U(\alpha^*) \cdot \nabla \tilde{f}_{\alpha^*}^U(\mathbf{x}) = \lambda^L(\alpha^*) \cdot \mathbf{c}_{\alpha^*}^L + \lambda^U(\alpha^*) \cdot \mathbf{c}_{\alpha^*}^U. \tag{9}$$

We consider the following constrained optimization problem:

$$\begin{array}{lll} \text{(P1)} & \min & f(\mathbf{x}) \\ & \text{subject to} & g_j(\mathbf{x}) = a_{j1}x_1 + \cdots + a_{jn}x_n + b_j \leq 0, j = 1, \ldots, m, \\ & & g_{m+k}(\mathbf{x}) = -x_k \leq 0, k = 1, \ldots, n, \end{array}$$

where f is defined in (7). Then problems (P1) and (FLP1) have identical feasible regions. Since conditions (i) and (ii) of this theorem are satisfied for all $\alpha \in [0, 1]$, and $\mathbf{a}_j = \nabla g_j(\mathbf{x})$ for all $j = 1, \ldots, m$, according to (9) and for any fixed $\alpha^* \in [0, 1]$, we can obtain the following two new conditions by letting $\mu_{j\alpha^*} = \mu_j(\alpha^*) \geq 0$ for $j = 1, \ldots, m$, $g_{m+k}(\mathbf{x}) = -x_k$ and $\lambda_{k\alpha^*} = \lambda_k(\alpha^*)$ for $k = 1, \ldots, n$:

(i′) $\nabla f(\mathbf{x}^*) + \sum_{j=1}^m \mu_{j\alpha^*} \cdot \nabla g_j(\mathbf{x}^*) + \sum_{k=1}^n \lambda_{k\alpha^*} \cdot \nabla g_{m+k}(\mathbf{x}^*) = \lambda^L(\alpha^*) \cdot \mathbf{c}_{\alpha^*}^L + \lambda^U(\alpha^*) \cdot \mathbf{c}_{\alpha^*}^U + \sum_{j=1}^m \mu_j(\alpha^*) \cdot \mathbf{a}_j - \sum_{k=1}^n \lambda_k(\alpha^*) \cdot \mathbf{e}_k = \mathbf{0}$;
(ii′) $\mu_{j\alpha^*} \cdot g_j(\mathbf{x}^*) = 0 = \lambda_{k\alpha^*} \cdot g_{m+k}(\mathbf{x}^*)$ for all $j = 1, \ldots, m$ and all $k = 1, \ldots, n$.

Using Theorem 4.1, we see that the above conditions (i') and (ii') are the KKT conditions for problem (P1). Therefore, we conclude that $\mathbf{x}^*$ is an optimal solution of problem (P1), i.e., $f(\mathbf{x}^*) \leq f(\bar{\mathbf{x}})$, which contradicts (8). This completes the proof. ■

Remark 4.1. The positive real-valued functions λ^L and λ^U, and nonnegative real-valued functions μ_j and λ_k for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ can be constructed as follows. For any fixed $\alpha \in [0, 1]$, if there exist positive real numbers λ_α^L and λ_α^U, and nonnegative real numbers $\lambda_{k\alpha}$ and $\mu_{j\alpha}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that the following conditions are satisfied:

(a) $\lambda_\alpha^L \cdot \mathbf{c}_\alpha^L + \lambda_\alpha^U \cdot \mathbf{c}_\alpha^U + \sum_{j=1}^m \mu_{j\alpha} \cdot \mathbf{a}_j - \sum_{k=1}^n \lambda_{k\alpha} \cdot \mathbf{e}_k = \mathbf{0}$;
(b) $\mu_{j\alpha} \cdot g_j(\mathbf{x}^*) = 0 = \lambda_{k\alpha} \cdot x_k^*$ for all $j = 1, \ldots, m$ and $k = 1, \ldots, n$,

then we can define the positive real-valued functions $\lambda^L(\alpha) = \lambda_\alpha^L$ and $\lambda^U(\alpha) = \lambda_\alpha^U$ for all $\alpha \in [0, 1]$, and the nonnegative real-valued functions $\mu_j(\alpha) = \mu_{j\alpha}$ and $\lambda_k(\alpha) = \lambda_{k\alpha}$ for all $\alpha \in [0, 1]$, all $j = 1, \ldots, m$ and all $k = 1, \ldots, n$. Therefore, if the above conditions (a) and (b) are satisfied for all $\alpha \in [0, 1]$, then $\mathbf{x}^*$ is a nondominated type-I solution of problem (FLP1) by Theorem 4.2.

Theorem 4.3. *Let $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ be a feasible solution of problem* (FLP1). *If there exist positive real numbers λ^L and λ^U, nonnegative real numbers μ_j and λ_k for $j = 1, \ldots, m$ and $k = 1, \ldots, n$, and $\alpha^* \in [0, 1]$ such that the following conditions are satisfied:*

(i) $\lambda^L \cdot \mathbf{c}_{\alpha^*}^L + \lambda^U \cdot \mathbf{c}_{\alpha^*}^U + \sum_{j=1}^m \mu_j \cdot \mathbf{a}_j - \sum_{k=1}^n \lambda_k \cdot \mathbf{e}_k = \mathbf{0}$;
(ii) $\mu_j \cdot g_j(\mathbf{x}^*) = 0 = \lambda_k \cdot x_k^*$ *for all* $j = 1, \ldots, m$ *and all* $k = 1, \ldots, n$,

then $\mathbf{x}^*$ is a nondominated type-II solution of problem (FLP1).

Proof. We are going to prove this result by contradiction. Suppose that conditions (i) and (ii) are satisfied and $\mathbf{x}^*$ is not a nondominated type-II solution. Then there exists a feasible solution $\bar{\mathbf{x}}$ such that $\tilde{f}(\bar{\mathbf{x}}) \prec_{II} \tilde{f}(\mathbf{x}^*)$, i.e., (3) is satisfied for all $\alpha \in [0, 1]$. For α^* in conditions (i) and (ii), we can define a real-valued function

$$f(\mathbf{x}) = (\lambda^L \cdot \tilde{c}_{1\alpha^*}^L + \lambda^U \cdot \tilde{c}_{1\alpha^*}^U) \cdot x_1 + \cdots + (\lambda^L \cdot \tilde{c}_{n\alpha^*}^L + \lambda^U \cdot \tilde{c}_{n\alpha^*}^U) \cdot x_n. \tag{10}$$

Then we see that

$$f(\mathbf{x}) = \lambda^L \cdot \tilde{f}_{\alpha^*}^L(\mathbf{x}) + \lambda^U \cdot \tilde{f}_{\alpha^*}^U(\mathbf{x}) \tag{11}$$

and

$$\nabla f(\mathbf{x}) = \lambda^L \cdot \nabla \tilde{f}_{\alpha^*}^L(\mathbf{x}) + \lambda^U \cdot \nabla \tilde{f}_{\alpha^*}^U(\mathbf{x}) = \lambda^L \cdot \mathbf{c}_{\alpha^*}^L + \lambda^U \cdot \mathbf{c}_{\alpha^*}^U. \tag{12}$$

Now we consider the constrained optimization problem (P1) as in the proof of Theorem 4.2. Then we can obtain the following two new conditions:

(i′) $\nabla f(\mathbf{x}^*) + \sum_{j=1}^m \mu_j \cdot \nabla g_j(\mathbf{x}^*) + \sum_{k=1}^n \lambda_k \cdot \nabla g_{m+k}(\mathbf{x}^*) = \lambda^L \cdot \mathbf{c}_{\alpha^*}^L + \lambda^U \cdot \mathbf{c}_{\alpha^*}^U + \sum_{j=1}^m \mu_j \cdot \mathbf{a}_j - \sum_{k=1}^n \lambda_k \cdot \mathbf{e}_k = \mathbf{0}$;
(ii′) $\mu_j \cdot g_j(\mathbf{x}^*) = 0 = \lambda_k \cdot g_{m+k}(\mathbf{x}^*)$ for all $j = 1, \ldots, m$ and all $k = 1, \ldots, n$.

The remaining proof follows from the similar arguments of Theorem 4.2. ■

We are going to present another type of optimality conditions for a nondominated type-II solution without considering α^*.

Definition 4.1. Let $\tilde{a}$ be a fuzzy number. We say that $\tilde{a}$ is a *canonical fuzzy number* if the functions $\eta_1(\alpha) = \tilde{a}_\alpha^L$ and $\eta_2(\alpha) = \tilde{a}_\alpha^U$ are continuous on [0,1].

Remark 4.2. Let $\tilde{a}$ be a fuzzy number and its membership function be strictly increasing on the interval $\left[\tilde{a}_0^L, \tilde{a}_1^L\right]$ and strictly decreasing on the interval $\left[\tilde{a}_1^U, \tilde{a}_0^U\right]$. Then, from the fact of strict monotonicity, $\tilde{a}_\alpha^L$ and $\tilde{a}_\alpha^U$ are continuous functions with respect to the variable α on [0,1]. This shows that $\tilde{a}$ is a canonical fuzzy number.

Let $\mathbf{f} : [a, b] \to \mathbb{R}^n$ be a vector-valued function defined on the closed interval such that each component f_i, $i = 1, \ldots, n$, is continuous on $[a, b]$. Then the Riemann integral of $\mathbf{f}$ on $[a, b]$ is defined to be the Riemann integral of each component f_i on $[a, b]$. More precisely, we have

$$\int_a^b \mathbf{f}(x)\mathrm{d}x = \left[\int_a^b f_1(x)\mathrm{d}x, \ldots, \int_a^b f_n(x)\mathrm{d}x\right].$$

Now we are in a position to present another type of optimality conditions by considering the Riemann integral.

Theorem 4.4. *Suppose that the fuzzy coefficients $\tilde{c}_i$ for $i = 1, \ldots, n$ in the fuzzy-valued objective function $\tilde{f}$ are now assumed to be canonical fuzzy numbers. Let $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ be a feasible solution of problem* (FLP1). *If there exist positive real numbers λ^L and λ^U, and nonnegative real numbers μ_j and λ_k for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that the following conditions are satisfied:*

(i) $\lambda^L \cdot \int_0^1 \mathbf{c}_\alpha^L \mathrm{d}\alpha + \lambda^U \cdot \int_0^1 \mathbf{c}_\alpha^U \mathrm{d}\alpha + \sum_{j=1}^m \mu_j \cdot \mathbf{a}_j - \sum_{k=1}^n \lambda_k \cdot \mathbf{e}_k = \mathbf{0}$;
(ii) $\mu_j \cdot g_j(\mathbf{x}^*) = 0 = \lambda_k \cdot x_k^*$ *for all* $j = 1, \ldots, m$ *and all* $k = 1, \ldots, n$,

then $\mathbf{x}^*$ is a nondominated type-II solution of problem (FLP1).

Proof. For any fixed $\mathbf{x}$, since $\tilde{f}_\alpha^L(\mathbf{x})$ and $\tilde{f}_\alpha^U(\mathbf{x})$ are continuous on [0,1] with respect to the variable α by definition, they will be Riemann integrable on [0,1] with respect to α. Therefore, we can define a real-valued function as follows:

$$f(\mathbf{x}) = \lambda^L \cdot \int_0^1 \tilde{f}_\alpha^L(\mathbf{x})\mathrm{d}\alpha + \lambda^U \cdot \int_0^1 \tilde{f}_\alpha^U(\mathbf{x})\mathrm{d}\alpha. \tag{13}$$

For any fixed α, since $\tilde{f}_\alpha^L(\mathbf{x})$ and $\tilde{f}_\alpha^U(\mathbf{x})$ are linear functions, that is, they are continuously differentiable, by Rudin [32, Theorem 9.42], we have

$$\nabla f(\mathbf{x}) = \lambda^L \cdot \int_0^1 \nabla \tilde{f}_\alpha^L(\mathbf{x})\mathrm{d}\alpha + \lambda^U \cdot \int_0^1 \nabla \tilde{f}_\alpha^U(\mathbf{x})\mathrm{d}\alpha = \lambda^L \cdot \int_0^1 \mathbf{c}_\alpha^L \mathrm{d}\alpha + \lambda^U \cdot \int_0^1 \mathbf{c}_\alpha^U \mathrm{d}\alpha. \tag{14}$$

It also says that condition (i) of this theorem is well defined. We are going to prove this result by contradiction. Suppose that $\mathbf{x}^*$ is not a nondominated type-II solution. Then there exists a feasible solution $\bar{\mathbf{x}}$ such that $\tilde{f}(\bar{\mathbf{x}}) \prec_{II} \tilde{f}(\mathbf{x}^*)$, i.e., (3) is satisfied for all $\alpha \in [0, 1]$. Therefore, we have

$$\lambda^L \cdot \tilde{f}_\alpha^L(\bar{\mathbf{x}}) + \lambda^U \cdot \tilde{f}_\alpha^U(\bar{\mathbf{x}}) < \lambda^L \cdot \tilde{f}_\alpha^L(\mathbf{x}^*) + \lambda^U \cdot \tilde{f}_\alpha^U(\mathbf{x}^*)$$

for all $\alpha \in [0, 1]$ since $\lambda^L > 0$ and $\lambda^U > 0$. By taking integration with respect to α on [0,1] and using (13), we obtain $f(\bar{\mathbf{x}}) < f(\mathbf{x}^*)$. Now we consider the constrained optimization problem (P1) as in the proof of Theorem 4.2. Applying (14) to conditions (i) and (ii) of this theorem, we obtain the following two new conditions:

(i′) $\nabla f(\mathbf{x}^*) + \sum_{j=1}^m \mu_j \cdot \nabla g_j(\mathbf{x}^*) + \sum_{k=1}^n \lambda_k \cdot \nabla g_{m+k}(\mathbf{x}^*) = \mathbf{0}$;
(ii′) $\mu_j \cdot g_j(\mathbf{x}^*) = 0 = \lambda_k \cdot g_{m+k}(\mathbf{x}^*)$ for all $j = 1, \ldots, m$ and all $k = 1, \ldots, n$.

The remaining proof follows from the similar arguments of Theorem 4.2. ■

Remark 4.3. From the proof of Theorem 4.4, we see that if the function f in (13) is defined as

$$f(\mathbf{x}) = \int_0^1 \tilde{f}_\alpha^L(\mathbf{x})\mathrm{d}\alpha + \int_0^1 \tilde{f}_\alpha^U(\mathbf{x})\mathrm{d}\alpha,$$

then Theorem 4.4 still holds true if condition (i) is replaced by the following new condition:

$$\int_0^1 \mathbf{c}_\alpha^L \mathrm{d}\alpha + \int_0^1 \mathbf{c}_\alpha^U \mathrm{d}\alpha + \sum_{j=1}^m \mu_j \cdot \mathbf{a}_j - \sum_{k=1}^n \lambda_k \cdot \mathbf{e}_k = \mathbf{0}.$$

Now we are going to present the optimality conditions of problem (FLP1) in the fuzzy-valued form. We write $\tilde{\mathbf{0}} = (\tilde{0}, \ldots, \tilde{0})^T$. Let $\mathbf{x}$ be an n-vector in $\mathbb{R}^n$. Then the crisp vector $\tilde{1}_{\{\mathbf{x}\}}$ is defined as $\tilde{1}_{\{\mathbf{x}\}} = \left(\tilde{1}_{\{x_1\}}, \tilde{1}_{\{x_2\}}, \ldots, \tilde{1}_{\{x_n\}}\right)$. Let $\mathbf{a} = (a_1, \ldots, a_n)$ be an n-vector in $\mathbb{R}^n$. We say that $\mathbf{a}$ has the same sign if and only if $a_i \geq 0$ for all $i = 1, \ldots, n$ simultaneously, or $a_i < 0$ for all $i = 1, \ldots, n$ simultaneously (i.e., the components of vector $\mathbf{a}$ have the same sign). Or, equivalently, $\mathbf{a}$ has the same sign if and only if $\mathbf{a} \geq \mathbf{0}$ or $\mathbf{a} < \mathbf{0}$.

Theorem 4.5. *Let $\mathbf{x}^*$ be a feasible solution of problem* (FLP1). *Let $\tilde{\mathbf{c}} = (\tilde{c}_1, \ldots, \tilde{c}_n)^T$. We assume that each vector $\mathbf{a}_j = \nabla g_j(\mathbf{x})$ has the same sign for $j = 1, \ldots, m$. If there exist nonnegative fuzzy numbers $\tilde{\mu}_j, \tilde{\lambda}_k \in \mathcal{F}(\mathbb{R})$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that the following conditions are satisfied:*

(i) $\tilde{\mathbf{c}} \oplus \left[\bigoplus_{j=1}^m \left(\tilde{\mu}_j \otimes \tilde{1}_{\{\mathbf{a}_j\}}\right)\right] \oplus \left[\bigoplus_{k=1}^n \left(\tilde{\lambda}_k \otimes \tilde{1}_{\{-\mathbf{e}_k\}}\right)\right] = \tilde{\mathbf{0}}$;

(ii) $\tilde{\mu}_j \otimes \tilde{1}_{\{g_j(\mathbf{x}^*)\}} = \tilde{0} = \tilde{\lambda}_k \otimes \tilde{1}_{\{x_k^*\}}$ *for all* $j = 1, \ldots, m$ *and* $k = 1, \ldots, n$,

then $\mathbf{x}^*$ *is a nondominated type-I solution of problem* (FLP1).

Proof. We assume that conditions (i) and (ii) are satisfied. Let $I_+ \subseteq \{1, \ldots, m\}$ and $I_- \subseteq \{1, \ldots, m\}$ be the index sets defined by

$$I_+ = \{j : \mathbf{a}_j \geq \mathbf{0}\} \quad \text{and} \quad I_- = \{j : \mathbf{a}_j < \mathbf{0}\}.$$

Since

$$\tilde{1}_{\{\mathbf{a}_j\}} = \left(\tilde{1}_{\{a_{j1}\}}, \tilde{1}_{\{a_{j2}\}}, \ldots, \tilde{1}_{\{a_{jn}\}}\right)^T,$$

the ith component of the formula in condition (i) is given by

$$\tilde{c}_i \oplus \left[\bigoplus_{j=1}^{m} \left(\tilde{\mu}_j \otimes \tilde{1}_{\{a_{ji}\}}\right)\right] \oplus \left[\bigoplus_{k=1}^{n} \left(\tilde{\lambda}_k \otimes \tilde{1}_{\{-\delta_{ki}\}}\right)\right] = \tilde{0}, \tag{15}$$

where $\delta_{ki} = 1$ if $i = k$ and $\delta_{ki} = 0$ if $i \neq k$. Since $\tilde{\mu}_j$ and $\tilde{\lambda}_k$ are nonnegative fuzzy numbers for all $j = 1, \ldots, m$ and all $k = 1, \ldots, n$, we have that $(\tilde{\mu}_j)_\alpha^L = \tilde{\mu}_{j\alpha}^L$, $(\tilde{\mu}_j)_\alpha^U = \tilde{\mu}_{j\alpha}^U$, $(\tilde{\lambda}_j)_\alpha^L = \tilde{\lambda}_{j\alpha}^L$ and $(\tilde{\lambda}_j)_\alpha^U = \tilde{\lambda}_{j\alpha}^U$ are nonnegative real numbers by Remark 2.1 for all $\alpha \in [0, 1]$, all $j = 1, \ldots, m$ and all $k = 1, \ldots, n$. Taking the α-level set of (15) by using Proposition 2.1, we have

$$\tilde{c}_{i\alpha}^L + \sum_{j \in I_+} \tilde{\mu}_{j\alpha}^L \cdot a_{ji} + \sum_{j \in I_-} \tilde{\mu}_{j\alpha}^U \cdot a_{ji} - \sum_{k=1}^{n} \tilde{\lambda}_{k\alpha}^U \cdot \delta_{ki} = 0 = \tilde{c}_{i\alpha}^U + \sum_{j \in I_+} \tilde{\mu}_{j\alpha}^U \cdot a_{ji} + \sum_{j \in I_-} \tilde{\mu}_{j\alpha}^L \cdot a_{ji} - \sum_{k=1}^{n} \tilde{\lambda}_{k\alpha}^L \cdot \delta_{ki}$$

for all $\alpha \in [0, 1]$ and all $i = 1, \ldots, n$. Equivalently, in vector form, we have

$$\mathbf{c}_\alpha^L + \sum_{j \in I_+} \tilde{\mu}_{j\alpha}^L \cdot \mathbf{a}_j + \sum_{j \in I_-} \tilde{\mu}_{j\alpha}^U \cdot \mathbf{a}_j - \sum_{k=1}^{n} \tilde{\lambda}_{k\alpha}^U \cdot \mathbf{e}_k = \mathbf{0} = \mathbf{c}_\alpha^U + \sum_{j \in I_+} \tilde{\mu}_{j\alpha}^U \cdot \mathbf{a}_j + \sum_{j \in I_-} \tilde{\mu}_{j\alpha}^L \cdot \mathbf{a}_j - \sum_{k=1}^{n} \tilde{\lambda}_{k\alpha}^L \cdot \mathbf{e}_k$$

for all $\alpha \in [0, 1]$, which also implies, by adding them together,

$$\mathbf{c}_\alpha^L + \mathbf{c}_\alpha^U + \sum_{j=1}^{m} \mu_{j\alpha} \cdot \mathbf{a}_j - \sum_{k=1}^{n} \lambda_{k\alpha} \cdot \mathbf{e}_k = \mathbf{0} \tag{16}$$

for all $\alpha \in [0, 1]$, where $\mu_{j\alpha} = \tilde{\mu}_{j\alpha}^L + \tilde{\mu}_{j\alpha}^U$ and $\lambda_{k\alpha} = \tilde{\lambda}_{k\alpha}^L + \tilde{\lambda}_{k\alpha}^U$ are nonnegative real numbers for all $\alpha \in [0, 1]$, all $j = 1, \ldots, m$ and all $k = 1, \ldots, n$. We are going to prove this theorem by contradiction. Suppose that $\mathbf{x}^*$ is not a nondominated type-I solution. Then there exists a feasible solution $\bar{\mathbf{x}}$ such that $\tilde{f}(\bar{\mathbf{x}}) \prec_I \tilde{f}(\mathbf{x}^*)$, i.e., (3) is satisfied for some $\alpha^* \in [0, 1]$. We now define a real-valued function f as in (6) or in (7). Since $g_j(\mathbf{x}^*) \leq 0$ for all $j = 1, \ldots, m$ and $x_k^* \geq 0$ for all $k = 1, \ldots, n$, taking the α-level set of condition (ii) by using Proposition 2.1, we obtain that

$$\tilde{\mu}_{j\alpha}^U \cdot g_j(\mathbf{x}^*) = \left(\tilde{\mu}_j \otimes \tilde{1}_{\{g_j(\mathbf{x}^*)\}}\right)_\alpha^L = 0 = \left(\tilde{\mu}_j \otimes \tilde{1}_{\{g_j(\mathbf{x}^*)\}}\right)_\alpha^U = \tilde{\mu}_{j\alpha}^L \cdot g_j(\mathbf{x}^*)$$

for all $\alpha \in [0, 1]$ and all $j = 1, \ldots, m$ and

$$\tilde{\lambda}_{k\alpha}^L \cdot x_k^* = \left(\tilde{\lambda}_k \otimes \tilde{1}_{\{x_k^*\}}\right)_\alpha^L = 0 = \left(\tilde{\lambda}_k \otimes \tilde{1}_{\{x_k^*\}}\right)_\alpha^U = \tilde{\lambda}_{j\alpha}^U \cdot x_k^*$$

for all $\alpha \in [0, 1]$ and all $k = 1, \ldots, n$, which imply, by adding them together,

$$0 = \tilde{\mu}_{j\alpha}^L \cdot g_j(\mathbf{x}^*) + \tilde{\mu}_{j\alpha}^U \cdot g_j(\mathbf{x}^*) = \mu_{j\alpha} \cdot g_j(\mathbf{x}^*) \tag{17}$$

for all $\alpha \in [0, 1]$ and all $j = 1, \ldots, m$ and

$$0 = \tilde{\lambda}_{k\alpha}^L \cdot x_k^* + \tilde{\lambda}_{k\alpha}^U \cdot x_k^* = \lambda_{k\alpha} \cdot x_k^* \tag{18}$$

for all $\alpha \in [0, 1]$ and all $k = 1, \ldots, n$. Now we consider the constrained optimization problem (P1) as in the proof of Theorem 4.2. According to Eqs. (16)–(18) and (9), we obtain the following two new conditions:

(i′) $\nabla f(\mathbf{x}^*) + \sum_{j=1}^{m} \mu_{j\alpha^*} \cdot \nabla g_j(\mathbf{x}^*) + \sum_{k=1}^{n} \lambda_{k\alpha^*} \cdot \nabla g_{m+k}(\mathbf{x}^*) = \mathbf{0}$ (note that (16) is satisfied for all $\alpha \in [0, 1]$);

(ii′) $\mu_{j\alpha^*} \cdot g_j(\mathbf{x}^*) = 0 = \lambda_{k\alpha^*} \cdot g_{m+k}(\mathbf{x}^*)$ for all $j = 1, \ldots, m$ and all $k = 1, \ldots, n$ (note that (17) and (18) are satisfied for all $\alpha \in [0, 1]$).

Using Theorem 4.1, we see that $\mathbf{x}^*$ is an optimal solution of problem (P1) by regarding conditions (i') and (ii') as the KKT conditions, i.e., $f(\mathbf{x}^*) \leq f(\bar{\mathbf{x}})$, which contradicts (8). This completes the proof. ■

4.2. Fuzzy constraints

Now we are going to derive the optimality conditions for problem (FLP2) with fuzzy constraints. We recall that the fuzzy constraints are presented as

$$\tilde{g}_j(\mathbf{x}) = (\tilde{a}_{j1} \otimes \tilde{1}_{x_1}) \oplus \cdots \oplus (\tilde{a}_{jn} \otimes \tilde{1}_{x_n}) \oplus \tilde{b}_j$$

for $j = 1, \ldots, m$. Using Proposition 2.1, we obtain

$$\tilde{g}_{j\alpha}^L(\mathbf{x}) \equiv (\tilde{g}_j)_\alpha^L(\mathbf{x}) = \tilde{a}_{j1\alpha}^L \cdot x_1 + \cdots + \tilde{a}_{jn\alpha}^L \cdot x_n + \tilde{b}_{j\alpha}^L$$
$$\tilde{g}_{j\alpha}^U(\mathbf{x}) \equiv (\tilde{g}_j)_\alpha^U(\mathbf{x}) = \tilde{a}_{j1\alpha}^U \cdot x_1 + \cdots + \tilde{a}_{jn\alpha}^U \cdot x_n + \tilde{b}_{j\alpha}^U$$

for $\alpha \in [0, 1]$, where

$$\tilde{b}_{j\alpha}^L = (\tilde{b}_j)_\alpha^L, \tilde{b}_{j\alpha}^U = (\tilde{b}_j)_\alpha^U, \tilde{a}_{ji\alpha}^L = (\tilde{a}_{ji})_\alpha^L \quad \text{and} \quad \tilde{a}_{ji\alpha}^U = (\tilde{a}_{ji})_\alpha^U.$$

We also write

$$\mathbf{a}_{j\alpha}^L = \begin{bmatrix} \tilde{a}_{j1\alpha}^L \\ \vdots \\ \tilde{a}_{jn\alpha}^L \end{bmatrix} \quad \text{and} \quad \mathbf{a}_{j\alpha}^U = \begin{bmatrix} \tilde{a}_{j1\alpha}^U \\ \vdots \\ \tilde{a}_{jn\alpha}^U \end{bmatrix}.$$

Then $\mathbf{a}_{j\alpha}^L = \nabla \tilde{g}_{j\alpha}^L(\mathbf{x})$ and $\mathbf{a}_{j\alpha}^U = \nabla \tilde{g}_{j\alpha}^U(\mathbf{x})$. Now we are in a position to derive the optimality conditions of problem (FLP2).

Theorem 4.6. *Let* $\mathbf{x}^*$ *be a feasible solution of problem* (FLP2).
(A) *If there exist positive real-valued functions* λ^L *and* λ^U *defined on [0,1], and nonnegative real-valued functions* μ_j *and* λ_k *for* $j = 1, \ldots, m$ *and* $k = 1, \ldots, n$ *defined on [0,1] such that the following conditions are satisfied:*

(i) $\lambda^L(\alpha) \cdot \mathbf{c}_\alpha^L + \lambda^U(\alpha) \cdot \mathbf{c}_\alpha^U + \sum_{j=1}^{m} \mu_j(\alpha) \cdot \mathbf{a}_{j\alpha}^L - \sum_{k=1}^{n} \lambda_k(\alpha) \cdot \mathbf{e}_k = \mathbf{0}$ *for all* $\alpha \in [0, 1]$;

(ii) $\mu_j(\alpha) \cdot \tilde{g}_{j\alpha}^L(\mathbf{x}^*) = 0 = \lambda_k(\alpha) \cdot x_k^*$ *for all* $\alpha \in [0, 1]$, *all* $j = 1, \ldots, m$ *and all* $k = 1, \ldots, n$,

then $\mathbf{x}^*$ *is a nondominated type-I solution of problem* (FLP2).
(B) *If there exist positive real-valued functions* λ^L *and* λ^U *defined on [0,1], and nonnegative real-valued functions* μ_j *and* λ_k *for* $j = 1, \ldots, m$ *and* $k = 1, \ldots, n$ *defined on [0,1] such that the following conditions are satisfied:*

(iii) $\lambda^L(\alpha) \cdot \mathbf{c}_\alpha^L + \lambda^U(\alpha) \cdot \mathbf{c}_\alpha^U + \sum_{j=1}^{m} \mu_j(\alpha) \cdot \mathbf{a}_{j\alpha}^U - \sum_{k=1}^{n} \lambda_k(\alpha) \cdot \mathbf{e}_k = \mathbf{0}$ *for all* $\alpha \in [0, 1]$;

(iv) $\mu_j(\alpha) \cdot \tilde{g}_{j\alpha}^U(\mathbf{x}^*) = 0 = \lambda_k(\alpha) \cdot x_k^*$ *for all* $\alpha \in [0, 1]$, *all* $j = 1, \ldots, m$ *and all* $k = 1, \ldots, n$,

then $\mathbf{x}^*$ *is a nondominated type-I solution of problem* (FLP2).

Proof. (A) We are going to prove this result by contradiction. Suppose that conditions (i) and (ii) are satisfied and $\mathbf{x}^*$ is not a nondominated type-I solution. Then there exists a feasible solution $\bar{\mathbf{x}}$ such that $\tilde{f}(\bar{\mathbf{x}}) \prec_I \tilde{f}(\mathbf{x}^*)$, i.e., (3) is satisfied for some $\alpha^* \in [0, 1]$. We now define a real-valued function f as in (6) or in (7) and consider the following constrained optimization problem:

$$\begin{aligned} \text{(P2)} \qquad & \min \quad f(\mathbf{x}) \\ & \text{subject to} \quad \tilde{g}_{j\alpha^*}^L(\mathbf{x}) \leq 0, \; j = 1, \ldots, m \\ & \qquad\qquad\quad g_k(\mathbf{x}) = -x_k \leq 0, k = 1, \ldots, n. \end{aligned}$$

We see that $\tilde{g}_j(\mathbf{x}^*) \preceq \tilde{0}$ implies $\tilde{g}^L_{j\alpha}(\mathbf{x}^*) \leq 0$ and $\tilde{g}^U_{j\alpha}(\mathbf{x}^*) \leq 0$ for all $\alpha \in [0,1]$. This shows that if $\mathbf{x}^*$ is a feasible solution of problem (FLP2), then $\mathbf{x}^*$ is also a feasible solution of problem (P2). From (9) and letting $\mu_{j\alpha^*} = \mu_j(\alpha^*) \geq 0$ for $j = 1,\ldots,m$ and $\lambda_{k\alpha^*} = \lambda_k(\alpha^*)$ for $k = 1,\ldots,n$, we obtain the following two new conditions from conditions (i) and (ii):

(i′) $\nabla f(\mathbf{x}^*) + \sum_{j=1}^m \mu_{j\alpha^*} \cdot \nabla \tilde{g}^L_{j\alpha^*}(\mathbf{x}^*) + \sum_{k=1}^n \lambda_{k\alpha^*} \cdot \nabla g_k(\mathbf{x}^*) = \lambda^L(\alpha^*) \cdot \mathbf{c}^L_{\alpha^*} + \lambda^U(\alpha^*) \cdot \mathbf{c}^U_{\alpha^*} + \sum_{j=1}^m \mu_j(\alpha^*) \cdot \mathbf{a}^L_{j\alpha^*} - \sum_{k=1}^n \lambda_{k\alpha^*} \mathbf{e}_k = \mathbf{0}$;

(ii′) $\mu_{j\alpha^*} \cdot \tilde{g}^L_{j\alpha^*}(\mathbf{x}^*) = 0 = \lambda_{k\alpha^*} \cdot g_k(\mathbf{x}^*)$ for all $j = 1,\ldots,m$ and all $k = 1,\ldots,n$.

Using Theorem 4.1, we see that $\mathbf{x}^*$ is an optimal solution of problem (P2) by regarding the above conditions (i') and (ii') as the KKT conditions, i.e., $f(\mathbf{x}^*) \leq f(\bar{\mathbf{x}})$, which contradicts (8).

(B) We now consider the following constrained optimization problem:

$$
\begin{aligned}
(P2') \qquad & \min \quad f(\mathbf{x}) \\
& \text{subject to} \quad \tilde{g}^U_{j\alpha^*}(\mathbf{x}) \leq 0,\ j = 1,\ldots,m \\
& \qquad\qquad\quad g_k(\mathbf{x}) = -x_k \leq 0,\ k = 1,\ldots,n.
\end{aligned}
$$

Then we see that if $\mathbf{x}^*$ is a feasible solution of problem (FLP2), then $\mathbf{x}^*$ is also a feasible solution of problem (P2'). The above similar arguments can also be used. This completes the proof. ■

Theorem 4.7. *Let $\mathbf{x}^*$ be a feasible solution of problem* (FLP2).

(A) *If there exist positive real numbers λ^L and λ^U, nonnegative real numbers μ_j and λ_k for $j = 1,\ldots,m$ and $k = 1,\ldots,n$, and $\alpha^* \in [0,1]$ such that the following conditions are satisfied:*

(i) $\lambda^L \cdot \mathbf{c}^L_{\alpha^*} + \lambda^U \cdot \mathbf{c}^U_{\alpha^*} + \sum_{j=1}^m \mu_j \cdot \mathbf{a}^L_{j\alpha^*} - \sum_{k=1}^n \lambda_k \cdot \mathbf{e}_k = \mathbf{0}$;

(ii) $\mu_j \cdot \tilde{g}^L_{j\alpha^*}(\mathbf{x}^*) = 0 = \lambda_k \cdot x^*_k$ *for all $j = 1,\ldots,m$ and all $k = 1,\ldots,n$,*

then $\mathbf{x}^*$ is a nondominated type-II solution of problem (FLP2).

(B) *If there exist positive real numbers λ^L and λ^U, nonnegative real numbers μ_j and λ_k for $j = 1,\ldots,m$ and $k = 1,\ldots,n$, and $\alpha^* \in [0,1]$ such that the following conditions are satisfied:*

(iii) $\lambda^L \cdot \mathbf{c}^L_{\alpha^*} + \lambda^U \cdot \mathbf{c}^U_{\alpha^*} + \sum_{j=1}^m \mu_j \cdot \mathbf{a}^U_{j\alpha^*} - \sum_{k=1}^n \lambda_k \cdot \mathbf{e}_k = \mathbf{0}$;

(iv) $\mu_j \cdot \tilde{g}^U_{j\alpha^*}(\mathbf{x}^*) = 0 = \lambda_k \cdot x^*_k$ *for all $j = 1,\ldots,m$ and all $k = 1,\ldots,n$,*

then $\mathbf{x}^*$ is a nondominated type-II solution of problem (FLP2).

Proof. (A) We are going to prove this result by contradiction. Suppose that conditions (i) and (ii) are satisfied and $\mathbf{x}^*$ is not a nondominated type-II solution. Then there exists a feasible solution $\bar{\mathbf{x}}$ such that $\tilde{f}(\bar{\mathbf{x}}) \prec_{II} \tilde{f}(\mathbf{x}^*)$, i.e., (3) is satisfied for all $\alpha \in [0,1]$. For α^* in conditions (i) and (ii), we now define a real-valued function f as in (10) or in (11). Then we have $\nabla f(\mathbf{x}) = \lambda^L \cdot \mathbf{c}^L_{\alpha^*} + \lambda^U \cdot \mathbf{c}^U_{\alpha^*}$ from (12). Now we consider the constrained optimization problem (P2) as in the proof of Theorem 4.6. Then we can obtain the following two new conditions:

(i′) $\nabla f(\mathbf{x}^*) + \sum_{j=1}^m \mu_j \cdot \nabla \tilde{g}^L_{j\alpha^*}(\mathbf{x}^*) + \sum_{k=1}^n \lambda_k \cdot \nabla g_k(\mathbf{x}^*) = \lambda^L \cdot \mathbf{c}^L_{\alpha^*} + \lambda^U \cdot \mathbf{c}^U_{\alpha^*} + \sum_{j=1}^m \mu_j \cdot \mathbf{a}^L_{j\alpha^*} - \sum_{k=1}^n \lambda_k \cdot \mathbf{e}_k = \mathbf{0}$;

(ii′) $\mu_j \cdot \tilde{g}^L_{j\alpha^*}(\mathbf{x}^*) = 0 = \lambda_k \cdot g_k(\mathbf{x}^*)$ for all $j = 1,\ldots,m$ and all $k = 1,\ldots,n$.

The remaining proof follows from the similar arguments of Theorem 4.6.

(B) The above similar arguments can be used by considering problem $(P2')$. ■

Theorem 4.8. *Suppose that the fuzzy coefficients $\tilde{c}_i$ in the fuzzy-valued objective function $\tilde{f}$ and $\tilde{a}_{ji}$ in the fuzzy-valued constraint functions $\tilde{g}_j$ for $i = 1,\ldots,n$ and $j = 1,\ldots,m$ are now assumed to be canonical fuzzy numbers. Let $\mathbf{x}^*$ be a feasible solution of problem* (FLP2).

(A) *If there exist positive real numbers λ^L and λ^U, and nonnegative real numbers μ_j and λ_k for $j = 1,\ldots,m$ and $k = 1,\ldots,n$ such that the following conditions are satisfied:*

(i) $\lambda^L \cdot \int_0^1 \mathbf{c}^L_\alpha \mathrm{d}\alpha + \lambda^U \cdot \int_0^1 \mathbf{c}^U_\alpha \mathrm{d}\alpha + \sum_{j=1}^m \mu_j \cdot \int_0^1 \mathbf{a}^L_{j\alpha} \mathrm{d}\alpha - \sum_{k=1}^n \lambda_k \cdot \mathbf{e}_k = \mathbf{0}$;

(ii) $\mu_j \cdot \int_0^1 \tilde{g}^L_{j\alpha}(\mathbf{x}^*) \mathrm{d}\alpha = 0 = \lambda_k \cdot x^*_k$ *for all $j = 1,\ldots,m$ and all $k = 1,\ldots,n$,*

then $\mathbf{x}^*$ is a nondominated type-II solution of problem (FLP2).

(B) *If there exist positive real numbers* λ^L *and* λ^U, *and nonnegative real numbers* μ_j *and* λ_k *for* $j = 1, \ldots, m$ *and* $k = 1, \ldots, n$ *such that the following conditions are satisfied:*

(iii) $\lambda^L \cdot \int_0^1 \mathbf{c}_\alpha^L \mathrm{d}\alpha + \lambda^U \cdot \int_0^1 \mathbf{c}_\alpha^U \mathrm{d}\alpha + \sum_{j=1}^m \mu_j \cdot \int_0^1 \mathbf{a}_{j\alpha}^U \mathrm{d}\alpha - \sum_{k=1}^n \lambda_k \cdot \mathbf{e}_k = \mathbf{0}$;

(iv) $\mu_j \cdot \int_0^1 \tilde{g}_{j\alpha}^U(\mathbf{x}^*)\mathrm{d}\alpha = 0 = \lambda_k \cdot x_k^*$ *for all* $j = 1, \ldots, m$ *and all* $k = 1, \ldots, n$,

then $\mathbf{x}^*$ *is a nondominated type-II solution of problem* (FLP2).

Proof. (A) We consider $f(\mathbf{x})$ and $\nabla f(\mathbf{x})$ as given in (13) and (14). We also consider the following constrained optimization problem:

$$\begin{array}{lll}(\mathrm{P3}) & \min & f(\mathbf{x}) \\ & \text{subject to} & G_j^L(\mathbf{x}) \le 0,\ j = 1, \ldots, m \\ & & g_k(\mathbf{x}) = -x_k \le 0, k = 1, \ldots, n\end{array}$$

where

$$G_j^L(\mathbf{x}) = \int_0^1 \tilde{g}_{j\alpha}^L(\mathbf{x})\mathrm{d}\alpha,$$

since, for any fixed $\mathbf{x}$, $\tilde{g}_{j\alpha}^L(\mathbf{x})$ is also Riemann integrable on [0,1] with respect to the variable α by definition. Then we have

$$\nabla G_j^L(\mathbf{x}) = \int_0^1 \nabla \tilde{g}_{j\alpha}^L(\mathbf{x})\mathrm{d}\alpha = \int_0^1 \mathbf{a}_{j\alpha}^L \mathrm{d}\alpha, \tag{19}$$

since, for any fixed α, $\tilde{g}_{j\alpha}^L(\mathbf{x})$ is a linear function, i.e., continuously differentiable. Let $\mathbf{x}^*$ be a feasible solution of problem (FOP2), i.e., $\tilde{g}_{j\alpha}^L(\mathbf{x}^*) \le 0$ for all $\alpha \in [0, 1]$, which implies $G_j^L(\mathbf{x}^*) \le 0$ by taking integration on [0,1] with respect to α. This shows that $\mathbf{x}^*$ is also a feasible solution of problem (P3). We are going to prove this result by contradiction. Suppose that $\mathbf{x}^*$ is not a nondominated type-II solution. Then there exists a feasible solution $\bar{\mathbf{x}}$ such that $\tilde{f}(\bar{\mathbf{x}}) \prec_{II} \tilde{f}(\mathbf{x}^*)$, i.e., (3) is satisfied for all $\alpha \in [0, 1]$. Applying (14) and (19) to conditions (i) and (ii) of this theorem, we obtain the following two new conditions:

(i$'$) $\nabla f(\mathbf{x}^*) + \sum_{j=1}^m \mu_j \cdot \nabla G_j^L(\mathbf{x}^*) + \sum_{k=1}^n \lambda_k \cdot \nabla g_k(\mathbf{x}^*) = \mathbf{0}$;

(ii$'$) $\mu_j \cdot G_j^L(\mathbf{x}^*) = 0 = \lambda_k \cdot g_k(\mathbf{x}^*)$ for all $j = 1, \ldots, m$ and $k = 1, \ldots, n$.

The above two conditions can be regarded as the KKT conditions of problem (P3). The remaining proof follows from the similar arguments of Theorem 4.6.

(B) We consider the following constrained optimization problem:

$$\begin{array}{lll}(P3') & \min & f(\mathbf{x}) \\ & \text{subject to} & G_j^U(\mathbf{x}) \le 0,\ j = 1, \ldots, m \\ & & g_k(\mathbf{x}) = -x_k \le 0, k = 1, \ldots, n\end{array}$$

where

$$G_j^U(\mathbf{x}) = \int_0^1 \tilde{g}_{j\alpha}^U(\mathbf{x})\mathrm{d}\alpha,$$

since, for any fixed $\mathbf{x}$, $\tilde{g}_{j\alpha}^U(\mathbf{x})$ is also Riemann integrable on [0,1] with respect to the variable α by definition. Then we can also show that if $\mathbf{x}^*$ is a feasible solution of problem (FOP2), then $\mathbf{x}^*$ is a feasible solution of problem (P3'). The above similar arguments can also be used. This completes the proof. ■

Next we are going to present the optimality conditions for problem (FLP2) in the fuzzy-valued form. Let $\tilde{\mathbf{a}} = (\tilde{a}_1, \ldots, \tilde{a}_n)$ be an n-vector consisting of fuzzy numbers $\tilde{a}_1, \ldots, \tilde{a}_n$. We say that $\tilde{\mathbf{a}}$ has the same sign if and only if, for any fixed $\alpha \in [0, 1]$, $\tilde{a}_{i\alpha}^L$, $i = 1, \ldots, n$, have the same sign simultaneously (i.e., $\tilde{a}_{i\alpha}^L \ge 0$ for all $i = 1, \ldots, n$, or $\tilde{a}_{i\alpha}^L < 0$ for all $i = 1, \ldots, n$), and $\tilde{a}_{i\alpha}^U$, $i = 1, \ldots, n$, have the same sign simultaneously.

Theorem 4.9. *Let $\mathbf{x}^*$ be a feasible solution of problem* (FLP2). *Let $\tilde{\mathbf{c}} = (\tilde{c}_1, \ldots, \tilde{c}_n)^T$ and $\tilde{\mathbf{a}}_j = (\tilde{a}_{j1}, \ldots, \tilde{a}_{jn})^T$ for $j = 1, \ldots, m$. We assume that each n-vector $\tilde{\mathbf{a}}_j$ has the same sign for $j = 1, \ldots, m$. If there exist nonnegative fuzzy numbers $\tilde{\mu}_j$ and $\tilde{\lambda}_k$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that the following conditions are satisfied:*

(i) $\tilde{\mathbf{c}} \oplus \left[\bigoplus_{j=1}^m \left(\tilde{\mu}_j \otimes \tilde{\mathbf{a}}_j\right)\right] \oplus \left[\bigoplus_{k=1}^n \left(\tilde{\lambda}_k \otimes \tilde{1}_{\{-\mathbf{e}_k\}}\right)\right] = \tilde{\mathbf{0}}$;

(ii) $\tilde{\mu}_j \otimes \tilde{g}_j(\mathbf{x}^*) = \tilde{0} = \tilde{\lambda}_k \otimes \tilde{1}_{\{x_k^*\}}$ *for all $j = 1, \ldots, m$ and $k = 1, \ldots, n$,*

then $\mathbf{x}^*$ is a nondominated type-I solution of problem (FLP2).

Proof. We assume that conditions (i) and (ii) are satisfied. The ith component of the formula in condition (i) is given by

$$\tilde{c}_i \oplus \left[\bigoplus_{j=1}^m \left(\tilde{\mu}_j \otimes \tilde{a}_{ji}\right)\right] \oplus \left[\bigoplus_{k=1}^n \left(\tilde{\lambda}_k \otimes \tilde{1}_{\{-\delta_{ki}\}}\right)\right] = \tilde{0}, \tag{20}$$

where $\delta_{ki} = 1$ if $i = k$ and $\delta_{ki} = 0$ if $i \neq k$. Since $\tilde{\mu}_j$ and $\tilde{\lambda}_k$ are nonnegative for all $j = 1, \ldots, m$ and all $k = 1, \ldots, n$, taking the α-level set of (20) by using Proposition 2.1 and Remark 2.1, we then have

$$\tilde{c}_{i\alpha}^L + \sum_{j=1}^m \min\left\{\tilde{\mu}_{j\alpha}^L \tilde{a}_{ji\alpha}^L, \tilde{\mu}_{j\alpha}^U \tilde{a}_{ji\alpha}^L\right\} - \sum_{k=1}^n \tilde{\lambda}_{k\alpha}^U \delta_{ki} = 0 = \tilde{c}_{i\alpha}^U + \sum_{j=1}^m \max\left\{\tilde{\mu}_{j\alpha}^L \tilde{a}_{ji\alpha}^U, \tilde{\mu}_{j\alpha}^U \tilde{a}_{ji\alpha}^U\right\} - \sum_{k=1}^n \tilde{\lambda}_{k\alpha}^L \delta_{ki} \tag{21}$$

for all $\alpha \in [0, 1]$ and all $i = 1, \ldots, n$. Since each $\tilde{\mathbf{a}}_j$ has the same sign for $j = 1, \ldots, m$, we can adopt the following notations. Let $I_{\alpha+}^L$, $I_{\alpha-}^L$, $I_{\alpha+}^U$ and $I_{\alpha-}^U$ be index sets defined by

$$I_{\alpha+}^L = \left\{j : \tilde{a}_{ji\alpha}^L \geq 0\right\}, I_{\alpha-}^L = \left\{j : \tilde{a}_{ji\alpha}^L < 0\right\}, I_{\alpha+}^U = \left\{j : \tilde{a}_{ji\alpha}^U \geq 0\right\} \quad \text{and} \quad I_{\alpha-}^U = \left\{j : \tilde{a}_{ji\alpha}^U < 0\right\},$$

which are independent of index i. Then Eq. (21) can be rewritten as

$$\tilde{c}_{i\alpha}^L + \sum_{j \in I_{\alpha+}^L} \tilde{\mu}_{j\alpha}^L \tilde{a}_{ji\alpha}^L + \sum_{j \in I_{\alpha-}^L} \tilde{\mu}_{j\alpha}^U \tilde{a}_{ji\alpha}^L - \sum_{k=1}^n \tilde{\lambda}_{k\alpha}^U \delta_{ki} = 0 = \tilde{c}_{i\alpha}^U + \sum_{j \in I_{\alpha+}^U} \tilde{\mu}_{j\alpha}^U \tilde{a}_{ji\alpha}^U + \sum_{j \in I_{\alpha-}^U} \tilde{\mu}_{j\alpha}^L \tilde{a}_{ji\alpha}^U - \sum_{k=1}^n \tilde{\lambda}_{k\alpha}^L \delta_{ki}$$

for all $\alpha \in [0, 1]$ and all $i = 1, \ldots, n$; or, equivalently, in vector form,

$$\mathbf{c}_\alpha^L + \sum_{j \in I_{\alpha+}^L} \tilde{\mu}_{j\alpha}^L \mathbf{a}_{j\alpha}^L + \sum_{j \in I_{\alpha-}^L} \tilde{\mu}_{j\alpha}^U \mathbf{a}_{j\alpha}^L - \sum_{k=1}^n \tilde{\lambda}_{k\alpha}^U \mathbf{e}_k = \mathbf{0} = \mathbf{c}_\alpha^U + \sum_{j \in I_{\alpha+}^U} \tilde{\mu}_{j\alpha}^U \mathbf{a}_{j\alpha}^U + \sum_{j \in I_{\alpha-}^U} \tilde{\mu}_{j\alpha}^L \mathbf{a}_{j\alpha}^U - \sum_{k=1}^n \tilde{\lambda}_{k\alpha}^L \mathbf{e}_k$$

for all $\alpha \in [0, 1]$, which also implies, by adding them together,

$$\mathbf{c}_\alpha^L + \mathbf{c}_\alpha^U + \sum_{j=1}^m \eta_{j\alpha} \mathbf{a}_{j\alpha}^L + \sum_{j=1}^m \zeta_{j\alpha} \mathbf{a}_{j\alpha}^U - \sum_{k=1}^n \lambda_{k\alpha} \mathbf{e}_k = \mathbf{0} \tag{22}$$

for all $\alpha \in [0, 1]$, where

$$\eta_{j\alpha} = \begin{cases} \tilde{\mu}_{j\alpha}^L & \text{if } j \in I_{\alpha+}^L \\ \tilde{\mu}_{j\alpha}^U & \text{if } j \in I_{\alpha-}^L \end{cases}, \zeta_{j\alpha} = \begin{cases} \tilde{\mu}_{j\alpha}^L & \text{if } j \in I_{\alpha-}^U \\ \tilde{\mu}_{j\alpha}^U & \text{if } j \in I_{\alpha+}^U \end{cases} \quad \text{and} \quad \lambda_{k\alpha} = \tilde{\lambda}_{k\alpha}^L + \tilde{\lambda}_{k\alpha}^U$$

are all nonnegative real numbers for all $\alpha \in [0, 1]$ and all $j = 1, \ldots, m$. We are going to prove this theorem by contradiction. Suppose that $\mathbf{x}^*$ is not a nondominated type-I solution. Then there exists a feasible solution $\bar{\mathbf{x}}$ such that $\tilde{f}(\bar{\mathbf{x}}) \prec_I \tilde{f}(\mathbf{x}^*)$, i.e., (3) is satisfied for some $\alpha^* \in [0, 1]$. We now define a real-valued function f as in (6) or in (7). Since $\tilde{g}_j(\mathbf{x}^*) \preceq \tilde{0}$ and $\tilde{\mu}_j \succeq \tilde{0}$ for all $j = 1, \ldots, m$, from condition (ii) of this theorem and Proposition 2.2, we see that

$$\tilde{\mu}_{j\alpha}^L \cdot \tilde{g}_{j\alpha}^L(\mathbf{x}^*) = \tilde{\mu}_{j\alpha}^L \cdot \tilde{g}_{j\alpha}^U(\mathbf{x}^*) = \tilde{\mu}_{j\alpha}^U \cdot \tilde{g}_{j\alpha}^L(\mathbf{x}^*) = \tilde{\mu}_{j\alpha}^U \cdot \tilde{g}_{j\alpha}^U(\mathbf{x}^*) = 0$$

for all $\alpha \in [0, 1]$ and all $j = 1, \ldots, m$, and

$$\tilde{\lambda}_{k\alpha}^L \cdot x_k^* = \left(\tilde{\lambda}_k \otimes \tilde{1}_{\{x_k^*\}}\right)_\alpha^L = 0 = \left(\tilde{\lambda}_k \otimes \tilde{1}_{\{x_k^*\}}\right)_\alpha^U = \tilde{\lambda}_{j\alpha}^U \cdot x_k^*$$

for all $\alpha \in [0, 1]$ and all $k = 1, \ldots, n$, which imply

$$\eta_{j\alpha} \cdot \tilde{g}^L_{j\alpha}(\mathbf{x}^*) = 0 = \zeta_{j\alpha} \cdot \tilde{g}^U_{j\alpha}(\mathbf{x}^*) \tag{23}$$

for all $\alpha \in [0, 1]$ and all $j = 1, \ldots, m$, and

$$0 = \tilde{\lambda}^L_{k\alpha} \cdot x^*_k + \tilde{\lambda}^U_{k\alpha} \cdot x^*_k = \lambda_{k\alpha} \cdot x^*_k \tag{24}$$

for all $\alpha \in [0, 1]$ and all $k = 1, \ldots, n$. Now we consider the following constrained optimization problem:

$$\begin{array}{lll} \text{(P4)} & \min & f(\mathbf{x}) \\ & \text{subject to} & \tilde{g}^L_{j\alpha^*}(\mathbf{x}) \le 0,\ j = 1, \ldots, m \\ & & \tilde{g}^U_{j\alpha^*}(\mathbf{x}) \le 0,\ j = 1, \ldots, m \\ & & g_k(\mathbf{x}) = -x_k \le 0,\ k = 1, \ldots, n. \end{array}$$

Since $\tilde{g}_j(\mathbf{x}^*) \preceq \tilde{0}$ if and only if $\tilde{g}^L_{j\alpha}(\mathbf{x}^*) \le 0$ and $\tilde{g}^U_{j\alpha}(\mathbf{x}^*) \le 0$ for all $\alpha \in [0, 1]$, it shows that if $\mathbf{x}^*$ is a feasible solution of problem (FLP2) then $\mathbf{x}^*$ is also a feasible solution of problem (P4). The remaining proof is similar to the arguments of Theorem 4.6 by considering (22)–(24) and problem (P4). ■

5. Examples

Now we introduce the concept of triangular fuzzy number. The membership function of a triangular fuzzy number $\tilde{a}$ is defined by

$$\xi_{\tilde{a}}(r) = \begin{cases} (r - a^L)/(a - a^L) & \text{if } a^L \le r \le a \\ (a^U - r)/(a^U - a) & \text{if } a < r \le a^U \\ 0 & \text{otherwise,} \end{cases}$$

which is denoted by $\tilde{a} = (a^L, a, a^U)$. The graph of membership function of a triangular fuzzy number will be a triangle. The α-level set (a closed interval) of $\tilde{a}$ is then

$$\tilde{a}_\alpha = [(1 - \alpha)a^L + \alpha a, (1 - \alpha)a^U + \alpha a];$$

that is,

$$\tilde{a}^L_\alpha = (1 - \alpha)a^L + \alpha a \quad \text{and} \quad \tilde{a}^U_\alpha = (1 - \alpha)a^U + \alpha a. \tag{25}$$

Example 5.1. Now we consider the following optimization problem with crisp constraints:

$$\begin{array}{ll} \min & \left(\widetilde{-5} \otimes \tilde{1}_{\{x_1\}}\right) \oplus \left(\widetilde{-8} \otimes \tilde{1}_{\{x_2\}}\right) \oplus \left(\widetilde{-7} \otimes \tilde{1}_{\{x_3\}}\right) \oplus \left(\widetilde{-4} \otimes \tilde{1}_{\{x_4\}}\right) \oplus \left(\widetilde{-6} \otimes \tilde{1}_{\{x_5\}}\right) \\ \text{subject to} & 2x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5 \le 20 \\ & 3x_1 + 5x_2 + 4x_3 + 2x_4 + 4x_5 \le 30 \\ & x_1, x_2, x_3, x_4, x_5 \ge 0, \end{array}$$

where

$$\widetilde{-5} = (-6, -5, -3), \quad \widetilde{-8} = (-9, -8, -6), \quad \widetilde{-7} = (-8, -7, -4),$$
$$\widetilde{-4} = (-5, -4, -1), \quad \widetilde{-6} = (-7, -6, -5)$$

are triangular fuzzy numbers. We are going to apply Theorem 4.2 to obtain the nondominated type-I solution. Using (25), we obtain

$$\mathbf{c}^L_\alpha = \begin{bmatrix} -6 + \alpha \\ -9 + \alpha \\ -8 + \alpha \\ -5 + \alpha \\ -7 + \alpha \end{bmatrix}, \quad \mathbf{c}^U_\alpha = \begin{bmatrix} -3 - 2\alpha \\ -6 - 2\alpha \\ -4 - 3\alpha \\ -1 - 3\alpha \\ -5 - \alpha \end{bmatrix}, \quad \mathbf{a}_1 = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{a}_2 = \begin{bmatrix} 3 \\ 5 \\ 4 \\ 2 \\ 4 \end{bmatrix}. \tag{26}$$

Let us first solve the system of equations

$$\begin{cases} g_1(\mathbf{x}) = 2x_1 + 3x_2 + 3x_3 + 2x_4 + 2x_5 - 20 = 0 \\ g_2(\mathbf{x}) = 3x_1 + 5x_2 + 4x_3 + 2x_4 + 4x_5 - 30 = 0. \end{cases}$$

Then we obtain

$$\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) = (0, 5, 0, 2.5, 0).$$

We are going to check that this feasible solution $\mathbf{x}^*$ satisfies the optimality conditions (i) and (ii) in Theorem 4.2. From condition (ii) in Theorem 4.2, we see that $\lambda_2 = \lambda_4 = 0$ using $\mathbf{x}^*$. Now applying condition (i) to this point $\mathbf{x}^*$, we obtain

$$\lambda^L(\alpha) \cdot \mathbf{c}_\alpha^L + \lambda^L(\alpha) \cdot \mathbf{c}_\alpha^U + \sum_{j=1}^{2} \mu_j(\alpha) \cdot \mathbf{a}_j - \sum_{k=1}^{5} \lambda_k(\alpha) \cdot \mathbf{e}_k$$

$$= \begin{bmatrix} \lambda^L(\alpha) \cdot (-6+\alpha) + \lambda^U(\alpha) \cdot (-3-2\alpha) + 2\mu_1(\alpha) + 3\mu_2(\alpha) - \lambda_1(\alpha) \\ \lambda^L(\alpha) \cdot (-9+\alpha) + \lambda^U(\alpha) \cdot (-6-2\alpha) + 3\mu_1(\alpha) + 5\mu_2(\alpha) \\ \lambda^L(\alpha) \cdot (-8+\alpha) + \lambda^U(\alpha) \cdot (-4-3\alpha) + 3\mu_1(\alpha) + 4\mu_2(\alpha) - \lambda_3(\alpha) \\ \lambda^L(\alpha) \cdot (-5+\alpha) + \lambda^U(\alpha) \cdot (-1-3\alpha) + 2\mu_1(\alpha) + 2\mu_2(\alpha) \\ \lambda^L(\alpha) \cdot (-7+\alpha) + \lambda^U(\alpha) \cdot (-5-\alpha) + 2\mu_1(\alpha) + 4\mu_2(\alpha) - \lambda_5(\alpha) \end{bmatrix} = \mathbf{0}.$$

After some algebraic calculations, we can obtain the positive real-valued functions

$$\lambda^L(\alpha) = 1 = \lambda^U(\alpha)$$

and the nonnegative real-valued functions

$$\mu_1(\alpha) = 2\alpha, \quad \mu_2(\alpha) = 3 - \alpha \quad \text{and} \quad \lambda_k(\alpha) = 0 \text{ for } k = 1, \ldots, 5.$$

Therefore, $\mathbf{x}^* = (0, 5, 0, 2.5, 0)$ is a nondominated type-I solution.

Example 5.2. We are going to apply Theorem 4.4 to solve the same problem in Example 5.1 to obtain the nondominated type-II solution. It is easy to see that the triangular fuzzy numbers are also canonical fuzzy numbers. Therefore, the assumption in Theorem 4.4 is satisfied. We want to check that $\mathbf{x}^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) = (0, 5, 0, 2.5, 0)$ satisfies the optimality conditions in Theorem 4.4. From condition (ii), we see that $\lambda_2 = 0 = \lambda_4$. From (26), we obtain

$$\int_0^1 \mathbf{c}_\alpha^L \mathrm{d}\alpha = \begin{bmatrix} -5.5 \\ -8.5 \\ -7.5 \\ -4.5 \\ -6.5 \end{bmatrix} \quad \text{and} \quad \int_0^1 \mathbf{c}_\alpha^U \mathrm{d}\alpha = \begin{bmatrix} -4 \\ -7 \\ -5.5 \\ -2.5 \\ -5.5 \end{bmatrix}.$$

Applying condition (i) to $\mathbf{x}^*$, we obtain

$$\lambda^L \cdot \int_0^1 \mathbf{c}_\alpha^L \mathrm{d}\alpha + \lambda^U \cdot \int_0^1 \mathbf{c}_\alpha^U \mathrm{d}\alpha + \sum_{j=1}^{2} \mu_j \cdot \mathbf{a}_j - \sum_{k=1}^{5} \lambda_k \cdot \mathbf{e}_k$$

$$= \begin{bmatrix} -5.5\lambda^L - 4\lambda^U + 2\mu_1 + 3\mu_2 - \lambda_1 \\ -8.5\lambda^L - 7\lambda^U + 3\mu_1 + 5\mu_2 \\ -7.5\lambda^L - 5.5\lambda^U + 3\mu_1 + 4\mu_2 - \lambda_3 \\ -4.5\lambda^L - 2.5\lambda^U + 2\mu_1 + 2\mu_2 \\ -6.5\lambda^L - 5.5\lambda^U + 2\mu_1 + 4\mu_2 - \lambda_5 \end{bmatrix} = \mathbf{0}.$$

After some algebraic calculations, we obtain

$$\lambda^L = 1 = \lambda^U, \mu_1 = 1, \mu_2 = 2.5 \quad \text{and} \quad \lambda_k = 0 \text{ for } k = 1, \ldots, 5.$$

Therefore, Theorem 4.2 says that $\mathbf{x}^* = (0, 5, 0, 2.5, 0)$ is a nondominated type-II solution.

Example 5.3. Now we consider the following optimization problem with fuzzy constraints:

$$\begin{array}{ll} \min & \left(\widetilde{-4} \otimes \tilde{1}_{\{x_1\}}\right) \oplus \left(\widetilde{-3} \otimes \tilde{1}_{\{x_2\}}\right) \oplus \left(\widetilde{-6} \otimes \tilde{1}_{\{x_3\}}\right) \\ \text{subject to} & \left(\tilde{3} \otimes \tilde{1}_{\{x_1\}}\right) \oplus \left(\tilde{1} \otimes \tilde{1}_{\{x_2\}}\right) \oplus \left(\tilde{3} \otimes \tilde{1}_{\{x_3\}}\right) \oplus \widetilde{-30} \preceq \tilde{0} \\ & \left(\tilde{2} \otimes \tilde{1}_{\{x_1\}}\right) \oplus \left(\tilde{2} \otimes \tilde{1}_{\{x_2\}}\right) \oplus \left(\tilde{3} \otimes \tilde{1}_{\{x_3\}}\right) \oplus \widetilde{-40} \preceq \tilde{0} \\ & x_1, x_2, x_3 \geq 0, \end{array}$$

where

$$\begin{array}{lll} \widetilde{-4} = (-4.4, -4, -3.6), & \widetilde{-3} = (-3.3, -3, -2.7), & \widetilde{-6} = (-6.6, -6, -5.4) \\ \tilde{1} = (0.5, 1, 1.5), & \tilde{2} = (1.5, 2, 2.5), & \tilde{3} = (2.5, 3, 3.5) \\ \widetilde{-30} = (-32, -30, -28), & \widetilde{-40} = (-42, -40, -38). & \end{array}$$

We are going to apply Theorem 4.7 to obtain the nondominated type-II solution. Using (25), we obtain

$$\mathbf{c}_\alpha^L = \begin{bmatrix} -4.4 + 0.4\alpha \\ -3.3 + 0.3\alpha \\ -6.6 + 0.6\alpha \end{bmatrix}, \quad \mathbf{c}_\alpha^U = \begin{bmatrix} -3.6 - 0.4\alpha \\ -2.7 - 0.3\alpha \\ -5.4 - 0.6\alpha \end{bmatrix},$$

$$\mathbf{a}_{1\alpha}^L = \begin{bmatrix} 2.5 + 0.5\alpha \\ 0.5 + 0.5\alpha \\ 2.5 + 0.5\alpha \end{bmatrix}, \quad \mathbf{a}_{1\alpha}^U = \begin{bmatrix} 3.5 - 0.5\alpha \\ 1.5 - 0.5\alpha \\ 3.5 - 0.5\alpha \end{bmatrix}, \quad \mathbf{a}_{2\alpha}^L = \begin{bmatrix} 1.5 + 0.5\alpha \\ 1.5 + 0.5\alpha \\ 2.5 + 0.5\alpha \end{bmatrix} \quad \text{and} \quad \mathbf{a}_{2\alpha}^U = \begin{bmatrix} 2.5 - 0.5\alpha \\ 2.5 - 0.5\alpha \\ 3.5 - 0.5\alpha \end{bmatrix}.$$

Now we let

$$\tilde{g}_1(\mathbf{x}) = \left(\tilde{3} \otimes \tilde{1}_{\{x_1\}}\right) \oplus \left(\tilde{1} \otimes \tilde{1}_{\{x_2\}}\right) \oplus \left(\tilde{3} \otimes \tilde{1}_{\{x_3\}}\right) \oplus \widetilde{-30}$$

$$\tilde{g}_2(\mathbf{x}) = \left(\tilde{2} \otimes \tilde{1}_{\{x_1\}}\right) \oplus \left(\tilde{2} \otimes \tilde{1}_{\{x_2\}}\right) \oplus \left(\tilde{3} \otimes \tilde{1}_{\{x_3\}}\right) \oplus \widetilde{-40}.$$

Using Proposition 2.1, (25), we obtain

$$\tilde{g}_{1\alpha}^U(\mathbf{x}) = (3.5 - 0.5\alpha)x_1 + (1.5 - 0.5\alpha)x_2 + (3.5 - 0.5\alpha)x_3 - 28 - 2\alpha$$

$$\tilde{g}_{2\alpha}^U(\mathbf{x}) = (2.5 - 0.5\alpha)x_1 + (2.5 - 0.5\alpha)x_2 + (3.5 - 0.5\alpha)x_3 - 38 - 2\alpha.$$

Let us take $\alpha^* = 1$ and solve the following system of equations:

$$\begin{cases} \tilde{g}_{11}^U(\mathbf{x}) = 3x_1 + x_2 + 3x_3 - 30 = 0 \\ \tilde{g}_{21}^U(\mathbf{x}) = 2x_1 + 2x_2 + 3x_3 - 40 = 0. \end{cases}$$

Then we can obtain

$$\mathbf{x}^* = (x_1^*, x_2^*, x_3^*) = (0, 10, 20/3).$$

We are going to check that this feasible solution $\mathbf{x}^*$ satisfies the optimality conditions (iii) and (iv) in Theorem 4.7. It is easy to see that $\tilde{g}_{j\alpha}(\mathbf{x}^*) = 0$ for $\alpha^* = 1$ and $j = 1, 2$. From condition (iv) in Theorem 4.7, we have $\lambda_2 = \lambda_3 = 0$ using $\mathbf{x}^*$. Now applying condition (iii) to this point $\mathbf{x}^*$, we need to solve

$$\lambda^L \cdot \mathbf{c}_1^L + \lambda^U \cdot \mathbf{c}_1^U + \sum_{j=1}^{2} \mu_j \cdot \mathbf{a}_{j1}^U - \sum_{k=1}^{3} \lambda_k \cdot \mathbf{e}_k = \begin{bmatrix} -4\lambda^L - 4\lambda^U + 3\mu_1 + 2\mu_2 - \lambda_1 \\ -3\lambda^L - 3\lambda^U + \mu_1 + 2\mu_2 \\ -6\lambda^L - 6\lambda^U + 3\mu_1 + 3\mu_2 \end{bmatrix} = \mathbf{0}.$$

After some algebraic calculations, we can obtain

$$\lambda^L = 0.5 = \lambda^U \quad \text{and} \quad \mu_1 = \mu_2 = \lambda_1 = 1.$$

Therefore, $\mathbf{x}^* = (0, 10, 20/3)$ is a nondominated type-II solution.

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